A Generalization of the Preceding Paper "A Rabi Oscillation in Four and Five Level Systems"

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Abstract

In the preceding paper quant-ph/0312060 we considered a general model of an atom with n energy levels interacting with n–1 external laser fields and constructed a Rabi oscillation in the case of n =3, 4 and 5.

In the paper we present a systematic method getting along with computer to construct a Rabi oscillation in the general case.

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In the preceding paper [1] (and [2]) we considered a general model of an atom with n energy levels interacting with n–1 external laser fields and constructed a Rabi oscillation in the case of n = 3, 4 and 5. Concerning more realistic model on two level system see [3] and [4].

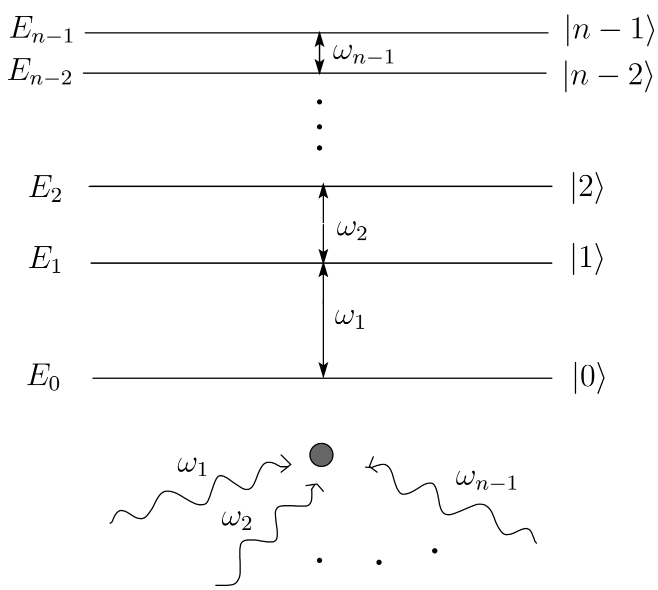
The purpose of this paper is to present a systematic method getting along with computer to construct a Rabi oscillation in the general case.

To begin with, let us make a brief review of the model. We consider an atom with n energy levels $\{(|k\rangle, E_k) \mid 0 \le k \le n-1\}$ which interacts with $n-1$ external fields. We set $\Delta_k \equiv E_k - E_0$ for $1 \le k \le n-1$ and assume the condition

$$E_1 - E_0 > E_2 - E_1 > \cdots > E_{n-1} - E_{n-2}$$

for simplicity.

We subject the atom to $n-1$ laser fields having the frequencies ω_k equal to the energy differences $\Delta_k - \Delta_{k-1} = E_k - E_{k-1}$. As an image see the following picture :



Then the evolution operator $U(t)$ defined by the Schrödinger equation

$$i\frac{d}{dt}U(t) = HU(t) \qquad (\hbar = 1),$$

where H is the Hamiltonian given in [1] (we don't repeat it here), is given by

$$U(t) = \mathrm{e}^{-itE_0} V^\dagger \mathrm{e}^{-itC}, \tag{1}$$

where $V = V(t)$ is

$$V = \begin{pmatrix} 1 & & & & & \\ & \mathrm{e}^{i(\omega_1 t+\phi_1)} & & & & \\ & & \mathrm{e}^{i(\omega_1 t+\omega_2 t+\phi_1+\phi_2)} & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & \mathrm{e}^{i\left(\sum_{k=1}^{n-2}\omega_k t+\sum_{k=1}^{n-2}\phi_k\right)} & \\ & & & & & \mathrm{e}^{i\left(\sum_{k=1}^{n-1}\omega_k t+\sum_{k=1}^{n-1}\phi_k\right)} \end{pmatrix}$$

and the constant matrix C consisting of all coupling constants

$$C \equiv C(g_1, g_2, \cdots, g_{n-1}) = \begin{pmatrix} 0 & g_1 & & & & & \\ g_1 & 0 & g_2 & & & & \\ & g_2 & 0 & g_3 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & g_{n-2} & 0 & g_{n-1} \\ & & & & & g_{n-1} & 0 \end{pmatrix}. \tag{2}$$

Therefore the remaining problem is to calculate the exponential e^{-itC}, which is however very hard.

In [1] we determined it by use of diagonalization method of matrices in the case of n = 3, 4 and 5. If all coupling constants are equal ($g_1 = g_2 = \cdots = g_{n-1}$) then the situation becomes very easy, see for example [5]. In the following we present a method to calculate e^{-itC} in the general case, which is universal in a sense.

Now let us review the formula in [6] within our necessity. Let A be a matrix in $M(n, \mathbf{C})$ and $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ the set of eigenvalues (containing multiplicities) of A. Then

the exponential e^{-itA} is given by the clear formula

$$\mathrm{e}^{-itA} = f_0(t)\mathbf{1}_n + f_1(t)A + f_2(t)A^2 + \cdots + f_{n-1}(t)A^{n-1} \tag{3}$$

with

$$f_l(t) = (-1)^{n+1}\sum_{k=1}^{n}\frac{(p_{n-l-1})_k\,\mathrm{e}^{-it\lambda_k}}{\prod_{j=1,j\neq k}^{n}(\lambda_j-\lambda_k)} \quad \text{for} \quad 0 \le l \le n-1. \tag{4}$$

Here $\{p_1, p_2, \cdots, p_{n-1}, p_n\}$ are a kind of fundamental symmetric polynomials of $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ and $\{(p_1)_k, (p_2)_k, \cdots, (p_{n-1})_k\}$ are them consisting of $\{\lambda_1, \cdots, \lambda_{k-1}, \lambda_{k+1}, \cdots, \lambda_n\}$. For example,

n=3 :

$$\mathrm{e}^{-itA} = f_0(t)\mathbf{1}_3 + f_1(t)A + f_2(t)A^2 \tag{5}$$

with

$$\begin{aligned}
f_0(t) &= \frac{\lambda_2\lambda_3\mathrm{e}^{-it\lambda_1}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} + \frac{\lambda_1\lambda_3\mathrm{e}^{-it\lambda_2}}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)} + \frac{\lambda_1\lambda_2\mathrm{e}^{-it\lambda_3}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)},\\
f_1(t) &= -\frac{(\lambda_2+\lambda_3)\mathrm{e}^{-it\lambda_1}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} - \frac{(\lambda_1+\lambda_3)\mathrm{e}^{-it\lambda_2}}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)} - \frac{(\lambda_1+\lambda_2)\mathrm{e}^{-it\lambda_3}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)},\\
f_2(t) &= \frac{\mathrm{e}^{-it\lambda_1}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} + \frac{\mathrm{e}^{-it\lambda_2}}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)} + \frac{\mathrm{e}^{-it\lambda_3}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}.
\end{aligned}$$

n=4 :

$$\mathrm{e}^{-itA} = f_0(t)\mathbf{1}_4 + f_1(t)A + f_2(t)A^2 + f_3(t)A^3 \tag{6}$$

with

$$\begin{aligned}
f_0(t) &= \frac{\lambda_2\lambda_3\lambda_4\mathrm{e}^{-it\lambda_1}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} + \frac{\lambda_1\lambda_3\lambda_4\mathrm{e}^{-it\lambda_2}}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)}\\
&\quad + \frac{\lambda_1\lambda_2\lambda_4\mathrm{e}^{-it\lambda_3}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)} + \frac{\lambda_1\lambda_2\lambda_3\mathrm{e}^{-it\lambda_4}}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)},\\
f_1(t) &= -\frac{(\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4)\mathrm{e}^{-it\lambda_1}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} - \frac{(\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_3\lambda_4)\mathrm{e}^{-it\lambda_2}}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)}\\
&\quad - \frac{(\lambda_1\lambda_2+\lambda_1\lambda_4+\lambda_2\lambda_4)\mathrm{e}^{-it\lambda_3}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)} - \frac{(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)\mathrm{e}^{-it\lambda_4}}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)},\\
f_2(t) &= \frac{(\lambda_2+\lambda_3+\lambda_4)\mathrm{e}^{-it\lambda_1}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} + \frac{(\lambda_1+\lambda_3+\lambda_4)\mathrm{e}^{-it\lambda_2}}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)}\\
&\quad + \frac{(\lambda_1+\lambda_2+\lambda_4)\mathrm{e}^{-it\lambda_3}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)} + \frac{(\lambda_1+\lambda_2+\lambda_3)\mathrm{e}^{-it\lambda_4}}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)},
\end{aligned}$$

$$f_3(t) = -\frac{e^{-it\lambda_1}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} - \frac{e^{-it\lambda_2}}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)}$$
$$-\frac{e^{-it\lambda_3}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)} - \frac{e^{-it\lambda_4}}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}.$$

It is notable that our formula (3) with (4) is convenient to use computer if the eigenvalues are known.

Next, what we do is to look for the eigenvalues of C in (2) in order to use the formula above. The characteristic polynomial of C is

$$f_n(\lambda) \equiv |\lambda \mathbf{1}_n - C| = \begin{vmatrix} \lambda & -g_1 & & & & & \\ -g_1 & \lambda & -g_2 & & & & \\ & -g_2 & \lambda & -g_3 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & -g_{n-2} & \lambda & -g_{n-1} \\ & & & & & -g_{n-1} & \lambda \end{vmatrix}. \tag{7}$$

By using the Laplace expansion of determinant it is easy to see

$$f_n(\lambda) = \lambda f_{n-1}(\lambda) - g_{n-1}^2 f_{n-2}(\lambda); \quad f_0(\lambda) = 1,\ f_1(\lambda) = \lambda. \tag{8}$$

For example,

$$\begin{aligned}
f_2(\lambda) &= \lambda^2 - g_1^2, \\
f_3(\lambda) &= \lambda\left(\lambda^2 - g_1^2 - g_2^2\right), \\
f_4(\lambda) &= \lambda^4 - (g_1^2+g_2^2+g_3^2)\lambda^2 + g_1^2 g_3^2, \\
f_5(\lambda) &= \lambda\left(\lambda^4 - (g_1^2+g_2^2+g_3^2+g_4^2)\lambda^2 + (g_1^2g_3^2+g_1^2g_4^2+g_2^2g_4^2)\right), \\
f_6(\lambda) &= \lambda^6 - (g_1^2+g_2^2+g_3^2+g_4^2+g_5^2)\lambda^4 + (g_1^2g_3^2+g_1^2g_4^2+g_1^2g_5^2+g_2^2g_4^2+g_2^2g_5^2+g_3^2g_5^2)\lambda^2 \\
&\quad -g_1^2g_3^2g_5^2 \\
f_7(\lambda) &= \lambda\Big(\lambda^6 - (g_1^2+g_2^2+g_3^2+g_4^2+g_5^2+g_6^2)\lambda^4 + (g_1^2g_3^2+g_1^2g_4^2+g_1^2g_5^2+g_1^2g_6^2+g_2^2g_4^2+ \\
&\quad g_2^2g_5^2+g_2^2g_6^6+g_3^2g_5^2+g_3^2g_6^2+g_4^2g_6^2)\lambda^2 - (g_1^2g_3^2g_5^2+g_1^2g_3^2g_6^2+g_1^2g_4^2g_6^2+g_2^2g_4^2g_6^2)\Big).
\end{aligned} \tag{9}$$

Now, let us look for the general form of the characteristic polynomial of $f_n(\lambda)$.

Main Result

(i) $n = 2m$:

$$f_{2m}(\lambda) = \lambda^{2m} - \phi_2\lambda^{2m-2} + \cdots + (-1)^k\phi_{2k}\lambda^{2m-2k} + \cdots + (-1)^{m-1}\phi_{2m-2}\lambda^2 + (-1)^m\phi_{2m} \quad (10)$$

with

$$\begin{aligned}
\phi_2 &= \sum_{i=1}^{2m-1} g_i^2, \\
\phi_{2k} &= \sum_{1\le i_1<i_2<i_3<\cdots<i_{k-1}<i_k;\ |i_1-i_2|\ge 2, |i_2-i_3|\ge 2, \cdots, |i_{k-1}-i_k|\ge 2}^{2m-1} g_{i_1}^2 g_{i_2}^2 \cdots g_{i_{k-1}}^2 g_{i_k}^2 \\
&\quad \text{for} \quad 2 \le k \le m-1.
\end{aligned} \quad (11)$$

(ii) $n = 2m+1$:

$$f_{2m+1}(\lambda) = \lambda\left\{\lambda^{2m} - \varphi_2\lambda^{2m-2} + \cdots + (-1)^k\varphi_{2k}\lambda^{2m-2k} + \cdots + (-1)^{m-1}\varphi_{2m-2}\lambda^2 + (-1)^m\varphi_{2m}\right\} \quad (12)$$

with

$$\begin{aligned}
\varphi_2 &= \sum_{i=1}^{2m} g_i^2, \\
\varphi_{2k} &= \sum_{1\le i_1<i_2<i_3<\cdots<i_{k-1}<i_k;\ |i_1-i_2|\ge 2, |i_2-i_3|\ge 2, \cdots, |i_{k-1}-i_k|\ge 2}^{2m} g_{i_1}^2 g_{i_2}^2 \cdots g_{i_{k-1}}^2 g_{i_k}^2 \\
&\quad \text{for} \quad 2 \le k \le m.
\end{aligned} \quad (13)$$

The key point of the proof is the following equations.

$$\begin{aligned}
\phi_{2k} &= \sum_{1\le i_1<i_2<i_3<\cdots<i_{k-1}<i_k;\ |i_1-i_2|\ge 2, |i_2-i_3|\ge 2, \cdots, |i_{k-1}-i_k|\ge 2}^{2m-1} g_{i_1}^2 g_{i_2}^2 \cdots g_{i_{k-1}}^2 g_{i_k}^2 \\
&= \sum_{1\le i_1<i_2<i_3<\cdots<i_{k-1};\ |i_1-i_2|\ge 2, |i_2-i_3|\ge 2, \cdots, |i_{k-2}-i_{k-1}|\ge 2}^{2m-3} g_{i_1}^2 g_{i_2}^2 \cdots g_{i_{k-1}}^2 g_{2m-1}^2 + \\
&\quad \sum_{1\le i_1<i_2<i_3<\cdots<i_{k-1}<i_k;\ |i_1-i_2|\ge 2, |i_2-i_3|\ge 2, \cdots, |i_{k-1}-i_k|\ge 2}^{2m-2} g_{i_1}^2 g_{i_2}^2 \cdots g_{i_{k-1}}^2 g_{i_k}^2
\end{aligned}$$

and

$$
\begin{aligned}
\varphi_{2k} &= \sum_{1\le i_1<i_2<i_3<\cdots<i_{k-1}<i_k;\ |i_1-i_2|\ge 2,|i_2-i_3|\ge 2,\cdots,|i_{k-1}-i_k|\ge 2}^{2m} g_{i_1}^2 g_{i_2}^2\cdots g_{i_{k-1}}^2 g_{i_k}^2 \\
&= \sum_{1\le i_1<i_2<i_3<\cdots<i_{k-1};\ |i_1-i_2|\ge 2,|i_2-i_3|\ge 2,\cdots,|i_{k-2}-i_{k-1}|\ge 2}^{2m-2} g_{i_1}^2 g_{i_2}^2\cdots g_{i_{k-1}}^2 g_{2m}^2 + \\
&\quad \sum_{1\le i_1<i_2<i_3<\cdots<i_{k-1}<i_k;\ |i_1-i_2|\ge 2,|i_2-i_3|\ge 2,\cdots,|i_{k-1}-i_k|\ge 2}^{2m-1} g_{i_1}^2 g_{i_2}^2\cdots g_{i_{k-1}}^2 g_{i_k}^2.
\end{aligned}
$$

For the proof we have only to use the mathematical induction. We leave the remaining part to readers.

A comment is in order. Since the matrix C in (2) is real symmetric (of course hermitian) its eigenvalues are all **real**.

Next, let us solve the characteristic polynomial of C. From (9)

$\underline{n=2}$:

$$\lambda_1 = g_1, \quad \lambda_2 = -g_1.$$

$\underline{n=3}$:

$$\lambda_1 = \sqrt{g_1^2+g_2^2}, \quad \lambda_2 = 0, \quad \lambda_3 = -\sqrt{g_1^2+g_2^2}.$$

$\underline{n=4}$:

$$\lambda_1 = \frac{\sqrt{A}+\sqrt{B}}{2}, \quad \lambda_2 = \frac{\sqrt{A}-\sqrt{B}}{2}, \quad \lambda_3 = -\frac{\sqrt{A}-\sqrt{B}}{2}, \quad \lambda_4 = -\frac{\sqrt{A}+\sqrt{B}}{2}$$

where

$$A = g_2^2 + (g_1+g_3)^2, \quad B = g_2^2 + (g_1-g_3)^2.$$

$\underline{n=5}$:

$$\lambda_1 = \frac{\sqrt{A}+\sqrt{B}}{2}, \quad \lambda_2 = \frac{\sqrt{A}-\sqrt{B}}{2}, \quad \lambda_3 = 0, \quad \lambda_4 = -\frac{\sqrt{A}-\sqrt{B}}{2}, \quad \lambda_5 = -\frac{\sqrt{A}+\sqrt{B}}{2}$$

where

$$A = g_1^2+g_2^2+g_3^2+g_4^2+2\sqrt{g_1^2g_3^2+g_1^2g_4^2+g_2^2g_4^2}, \quad B = g_1^2+g_2^2+g_3^2+g_4^2-2\sqrt{g_1^2g_3^2+g_1^2g_4^2+g_2^2g_4^2}.$$

$\underline{n=6}$:

By setting $\lambda^2 = x$ we can use the Cardano formula (see for example [7]). The solutions of the equation

$$x^3 - ax^2 + bx - c = 0$$

with

$$a = g_1^2 + g_2^2 + g_3^2 + g_4^2 + g_5^2,\ b = g_1^2 g_3^2 + g_1^2 g_4^2 + g_1^2 g_5^2 + g_2^2 g_4^2 + g_2^2 g_5^2 + g_3^2 g_5^2,\ c = g_1^2 g_3^2 g_5^2$$

are given by

$$x_1 = u_0 + v_0 + \frac{a}{3}, \quad x_2 = \sigma u_0 + \sigma^2 v_0 + \frac{a}{3}, \quad x_3 = \sigma^2 u_0 + \sigma v_0 + \frac{a}{3},$$

where $\sigma = \mathrm{e}^{2\pi i/3}$ and u_0, v_0 are each solution of the binomial equations

$$u^3 = \frac{-q + \sqrt{q^2 + 4p^3}}{2}, \quad v^3 = \frac{-q - \sqrt{q^2 + 4p^3}}{2}$$

with

$$p = \frac{b}{3} - \frac{a^2}{9}, \quad q = -c + \frac{ab}{3} - \frac{2a^3}{27}.$$

It is not difficult to show that we can choose $x_1 \geq x_2 \geq x_3 > 0$. Therefore the solutions are

$$\lambda_1 = \sqrt{x_1},\ \lambda_2 = \sqrt{x_2},\ \lambda_3 = \sqrt{x_3},\ \lambda_4 = -\sqrt{x_3},\ \lambda_5 = -\sqrt{x_2},\ \lambda_6 = -\sqrt{x_1}.$$

$\underline{n = 7}$:

In the case of $n = 6$ we have only to change a, b and c to

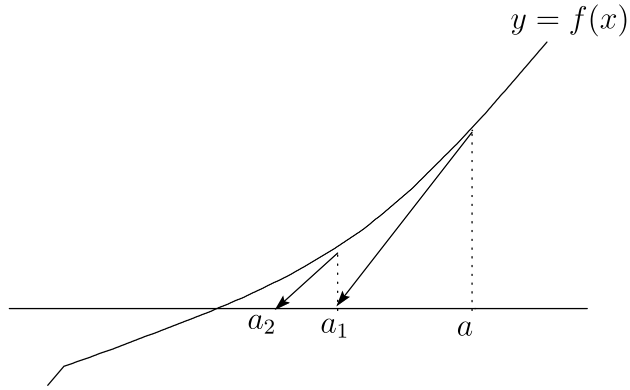
$$\begin{aligned} a &= g_1^2 + g_2^2 + g_3^2 + g_4^2 + g_5^2 + g_6^2, \\ b &= g_1^2 g_3^2 + g_1^2 g_4^2 + g_1^2 g_5^2 + g_1^2 g_6^2 + g_2^2 g_4^2 + g_2^2 g_5^2 + g_2^2 g_6^2 + g_3^2 g_5^2 + g_3^2 g_6^2 + g_4^2 g_6^2, \\ c &= g_1^2 g_3^2 g_5^2 + g_1^2 g_3^2 g_6^2 + g_1^2 g_4^2 g_6^2 + g_2^2 g_4^2 g_6^2 \end{aligned}$$

and obtain

$$\lambda_1 = \sqrt{x_1},\ \lambda_2 = \sqrt{x_2},\ \lambda_3 = \sqrt{x_3},\ \lambda_4 = 0,\ \lambda_5 = -\sqrt{x_3},\ \lambda_6 = -\sqrt{x_2},\ \lambda_7 = -\sqrt{x_1}.$$

For $n = 8$ and 9 we can in principle solve the characteristic polynomial (7) by use of the Ferrari or Euler formula (see [7]), which is of course very complicated. However, for

$n \geq 10$ it is impossible to obtain algebraic solutions of the characteristic polynomial by the famous Galois theory. Therefore we must appeal to some approximation method like the Newton's one (which is well–known), see the following picture.



Here for a given, the points a_1, a_2, etc are given by

$$a_1 = a - \frac{f(a)}{f'(a)},\ a_2 = a_1 - \frac{f(a_1)}{f'(a_1)},\ \cdots\ ,\ a_k = a_{k-1} - \frac{f(a_{k-1})}{f'(a_{k-1})},\ \cdots\ .$$

For an appropriate number n we have only to set $\lambda = a_n$. By changing a in the general case we obtain a set of approximate solutions $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$.

In this paper we generalized the result in [1] by making use of [6]. If we can find exact or approximate solutions of the characteristic polynomial (7) then we have the (exact or approximate) evolution operator by (3) and (4). That is, this means that a complicated unitary matrix in qudit theory was obtained. It may be possible to replace long quantum logic gates in qudit theory with few unitary matrices constructed in the paper, which will be discussed in another paper.

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